RUN-UP OF PERIODIC WAVES ON A STRAIGHT BEACH

ABSTRACT

Run-up heights of long linear periodic waves on a straight sloping beach are calculated. The incoming waves approach the beach at oblique angles of incidence. Resonance-like behaviour of the run-up height is found and related to coastal trapped waves.

1. INTRODUCTION

Several authors have considered the problem of run-up and reflection of waves on a sloping beach in the two dimensional case. Carrier & Greenspan (1958), Carrier (1966), Spielvogel (1976) and Gjevik & Pedersen (1981) have reported analytic run-up predictions based on non linear hydrostatic theory. Pedersen & Gjevik (1983) and Kim, Liu and Liggett (1983) have performed numerical calculations in which also dispersion effects are included. Experimental results are reported by Hall and Watts (1953).

To give good quantitative predictions, calculations of run-up heights in nature must be carried out with a bottom topography closely resembling the actual one. The study of some idealized cases will nevertheless be of interest. The investigation of special effects in such simple problems may provide us with qualitative information and rough estimates appliable also to more complex situations. In this report we study the run-up and reflection of periodic waves appraching a straight coastline at different angles of attack. We make the subject as simple as possible by using linear shallow water equations and a simple model of the bottom topography. Gjevik and Pedersen (1981) performed hydrostatic, two dimensional calculations closely resembling those which will be presented here. They found that inclusion of non linearity close to the shoreline did not affect the maximum run-up height. More precisely: The ratio between the maximum run-up height, R, and the amplitude, A, of the incoming wave was independent of A although other quantities like surface-shape and time for maximum run-up were not. This should justify that linear run-up calculations have some validity.

A lot of work has been done on the subject of propagation, refraction and reflection of waves in near shore regions without considering run-up on sloping beaches. J.W. Miles (1972) carried out an investigation of propagation and reflection of waves in coastal basins with geometries independent of the seaward coordinate. Although Miles had different boundary conditions at the shoreline, namely the zero flux condition of a vertical wall, and different bottom profiles his discussion of the importance of the trapped wave modes has relevance to the present report.

2. FORMULATION

We introduce Cartesian coordinates with the x_* and y_* axes in the undisturbed water surface. The equilibrium depth is denoted by h_* the surface elevation by η_* and the horizontal velocity by $\vec{v}_*=u_*\vec{i}+v_*\vec{j}$. The subscript $(*)$ indicates dimensional quantities. The linearized shallow water equations read:

$$\frac{\partial \eta_*}{\partial t_*} = -\nabla_*\cdot(h_*\vec{v}_*) = -h_*\nabla_*\cdot\vec{v}_* - \nabla_* h_*\cdot\vec{v}_*, \quad \frac{\partial \vec{v}_*}{\partial t_*} = -g\nabla_*\eta_* \tag{2.1}$$

$$\frac{\partial^2 \eta_*}{\partial t_*^2} - \nabla_*\cdot(gh_*\nabla_*\eta_*) = 0 \tag{2.2}$$

where $\nabla_* = \vec{i}\,\frac{\partial}{\partial x_*} + \vec{j}\,\frac{\partial}{\partial y_*}$, t_* denotes time and g is the acceleration of gravity. We will consider the simple topography defined by:

$$h_*(x_*,y_*) = \begin{cases} +\alpha x_* & \text{if} \quad 0<x_*<h_0/\alpha \\ h_0 & \text{if} \quad h_0/\alpha < x_* \end{cases} \tag{2.3}$$

We choose h_0 as the vertical length scale, h_0/x as the horizontal length scale and $\sqrt{h_0/(g\alpha^2)}$ as time scale. From (2.2) and (2.3) we get

$$\frac{\partial^2\eta}{\partial t^2} \; \frac{\partial}{\partial x}(x\frac{\partial\eta}{\partial x}) - x\frac{\partial^2\eta}{\partial y^2} = 0 \quad \text{for} \quad 0<x<1$$

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial x^2} - \frac{\partial^2\eta}{\partial y^2} = 0 \quad \text{for} \quad x>1 \tag{2.4}$$

where x, η, t and η are the dimensionless quantities corresponding to x_*, y_*, t_* and η_*. At $x=1$ we apply the conditions of continuous momentum and mass flux which after linearization turn out to be:

$$\eta = \text{continuous and } \frac{\partial\eta}{\partial x} = \text{continuous at} \quad x=1 \tag{2.5}$$

At the shoreline we have the condition

$$\frac{D(h+\eta)}{dt} = 0 \tag{2.6}$$

The linearized version of this equation is

$$\frac{\partial\eta}{\partial t} + \nabla h \cdot \vec{v} = 0 \tag{2.7}$$

which is automatically fulfilled because of the disappearance of the $\nabla \cdot \vec{v}$ term in the equation of continuity in (2.1) when $h=0$. When the time dependence is separated off, $h=0$ will give a singularity in the remaining equation in x and y. The equation (2.7) is thus replaced by the demand of η being finite for $x=0$.

In the offshore region $(x>1)$ we define a periodic incoming wave by:

$$\eta = A \sin(k(x-1)+\ell y+\omega t) \tag{2-8}$$

where the wave numbers k and ℓ and the frequency ω fit the dimensionless dispersion relation $\omega^2=k^2+\ell^2$. The angle of incidence of the incoming wave is

$$\phi = \arctan(k/\ell) \tag{2-9}$$

and the dimensionless wave length (scaled as the x coordinate) is

$$\lambda = 2\Pi/\omega = \lambda_{\star}\alpha/h_0 \tag{2-10}$$

Because the topography is independent of y the reflected wave in the offshore region has to be of the form:

$$\eta_r = D \sin\,(k(x-1)-\ell y-\omega t) + E \cos\,(k(x-1)-\ell y-\omega t) \tag{2-11}$$

In the near shore region we must have

$$\eta = (B \sin\,(\ell y+\omega t) + C \cos\,(\ell y+\omega t))Q(x;\omega^2,\ell^2) \tag{2-12}$$

where Q is the solution of the problem:

$$\frac{d}{dx}(x\frac{dQ}{dx}) + (\omega^2-\ell^2 x)Q = 0\ , \quad Q(0;\omega^2,\ell^2) = 1 \tag{2.13}$$

Because of the singularity of the differential equation, (2.13) determines Q uniquely. One of the solutions of the equation is infinite at $x=0$ and must be excluded. In some special cases the solution of (2.13) can be found in terms of simple analytic functions. For $\ell=0$ our problem is reduced to a two dimensional one and we have

$$Q(x;\omega^2,0) = J_0(2\omega\sqrt{x}) \tag{2.14}$$

where J_0 is the Besselfunction of zero order. (2.14) is easily verified by substitution. In this case Carrier (1966) and Gjevik & Pedersen (1981) have generalized the present calculations to include the effect of nonlinearity in the near shore region. When $\ell=\omega^2$ it is easily seen that

$$Q(x;\omega^2,\omega^4)=\ell^{-\omega^2 x} \tag{2.15}$$

For the general case the transformation

$$Q = \ell^{-\frac{1}{2}s}G(s)\ ; \quad s=2\ell x \tag{2-16}$$

gives an equation for G of the form:

$$s\,\frac{d^2G}{ds^2} + (b-s)\frac{dG}{ds} - aG = 0 \;;\quad G(s=0)=1 \tag{2.17}$$

where

$$a = \frac{1}{2}(1-\omega^2/\ell), \quad b=1 \tag{2.18}$$

The differential equation in (2.17) is known as Kummers equation, and the solution of this equation which also satisfies the initial condition $G(s=0)=1$ is the Kummers function: $M(a,b,s)$. Kummers function is described in Abramowitz and Stegun: "Handbook of Mathematical functions" from which we have adopted the notations. For the function Q, we may now write:

$$Q(x;\omega^2,\ell^2) = \ell^{-\ell x} M(\tfrac{1}{2}(1-\omega^2/\ell),1,2\ell x) \tag{2.19}$$

A consequence of (2.14) and (2.19) is that several asymptotic expansions are readily at hand. To get solutions valid also outside the range of this expansions we solve (2.14) directly by successive expansions in Taylor series. We first use the boundary condition and the differential equation to construct a Taylor series for Q at $x=0$. Then Q and $\frac{dQ}{dx}$ are calculated at some siutable point $x=x_j>0$ and a new Taylor series is formed at this point etc. For all cases reported in this paper Q is calculated with error less than 10^{-8}.

The matching conditions at $x=1$ give the following set of equations for the coefficients in (2.11) and (2.12):

$$\begin{aligned} BQ_1 &= A - D \;;\quad CQ_1 = E \\ BQ_1' &= kE \;;\quad CQ_1' = k(A+D) \end{aligned} \tag{2.20}$$

where $Q_1 = Q(1;\omega^2,\ell^2)$ and $Q_1' = \frac{dQ}{dx}(1;\omega^2,\ell^2)$. The solution of (2.20) is:

$$B = 2AQ_1/\gamma \ , \ C = 2AQ_1'/(k\gamma), \ D = A(Q_1'^2/k^2 - Q_1^2)/\gamma$$
$$E = 2AQ_1Q_1'/(k\gamma) \tag{2.21}$$

where $\gamma = Q_1^2 + (Q_1'/k)^2$.

For the maximum run-up height R we get the expression:

$$R(\omega^2, \ell^2) = (B^2+C^2)^{\frac{1}{2}} = 2A/\sqrt{\gamma} \tag{2.22}$$

When $\ell=0$ we have:

$$R(\omega^2, 0) = \frac{2A}{(J_0(2\omega)^2+J_1(2\omega)^2)^{\frac{1}{2}}} \tag{2.23}$$

For the special case $\ell=\omega^2$ (2.22) gives:

$$R(\omega^2, \omega^4) = 2 \cos \phi \ e^{\sin^2\phi} \tag{2.24}$$

where ϕ is as defined in (2.9).

For large ω use of asymptotic expansions for the Bessel functions on the right side of (2.23) gives:

$$R(\omega^2, 0) \sim 2A\sqrt{\Pi\omega} = 2\Pi A\sqrt{2/\lambda} \tag{2.25}$$

By use of (2.19) and differential formulas and asymptotic expansions of the Kummers function, a similar analysis may be performed in the general case when $\ell^2 \neq 0$. The result is an expression of the form:

$$R(\omega^2, \omega^2 \sin^2\phi) \sim 2A\sqrt{\omega}[L(\omega, \phi)+O(\frac{1}{\omega})] \tag{2.26}$$

where L is a complicated function of ω and ϕ which remains finite when $\omega \to \infty$. The equation (2.26) is not well fitted for discussion of the detailed behaviour of R but it shows that R is not bounded when $\omega \to \infty$.

For small ω the Taylor series for Q at $x=0$ converges rapidly. An expansion of R in powers of ω^2 is easily derived from this Taylor series. By retaining only the first two terms we get:

$$R = A[2+\{1 - \frac{1}{2}(\nu - \frac{1}{2}\nu^2)(1-\nu)^{-1}\}\omega^2+O(\omega^4)] \tag{2.27}$$

where

$$\nu = \sin^2\phi$$

For all values of ϕ, $R \to 2A$ as $\omega \to 0$. This limit corresponds to reflection from a rigid vertical wall. The coefficient before the ω^2 term changes from positive to negative values for $\phi \simeq 60.9^\circ$. Above this value for ϕ we will thus observe run-up values less than $2A$.

3. DISCUSSIONS

Relative run-up heights $F = R/A$ for angles of incidence, $\phi_1=0,30^\circ,40^\circ,\ldots,80^\circ$ are displayed in figure 2 as functions of λ. For the smaller values of ϕ, F is a decreasing function of λ and decreases slowly with ϕ. The difference between $F(\lambda,0)$ and $F(\lambda,30^\circ)$ is not more than about 8%. For larger values of ϕ and small λ the situation is more complex. One of the most striking feature by fig. 2 is the occurrence of the extemaes and inflection points. To get a better understanding of theese we turn to a discussion of the special case $\ell=0$. From (2.19) we get:

$$\frac{dF^2}{d(2\omega)} = \frac{1}{4\omega}J_1^2(2\omega)F^4 \tag{3.1}$$

which shows that $F(\lambda,0)$ is a monotonic decreasing function of λ and has points of inflection for which $J_1(2\omega_n) = 0$, $n=1,2,\ldots$. The smallest inflection point value is $\lambda = 3.28$. By use of asymptotic expansions for the Bessel functions the larger ones can be approximated by:

$$\lambda_n = 4/(n + \tfrac{1}{4}) \; ; \quad n=2,3,\ldots ,$$

From the equation (2.12) and (2.14) it is evident that for the inflection point values of ω we have:

$$\frac{\partial\eta}{\partial x} = u = 0 \quad \text{for} \quad x=1 \tag{3.2}$$

Hence the ω_n have physical interpretation as eigenfrequencies for the near shore region with a rigid wall at $x=1$.

This resonancelike effect must be expected to occur also for $\phi > 0$. We define $\lambda_n(\phi) = 2\Pi/\omega_n(\phi)$ by

$$\frac{dQ}{dx}(1;\omega_n^2,\omega_n^2 \sin^2\phi) = 0 \tag{3.3}$$

The λ_n are numbered in decreasing order. In figure 2 we have represented the points $\{\lambda_n(\phi),F(\lambda_n(\phi),\phi)\}$ (N=1,2) by the dotted lines. For all values of ϕ these lines intersect the undotted ones near the inflection points and the extremaes. The effect appears to be dominant for the highest values of ϕ. This is due to a transition towards "non decaying trapped" wave modes when $\phi \to \Pi/2$ which occurs only for certain values of ω.

For $\phi = \Pi/2$ there are no incoming or reflected waves. There is a class of trapped waves which for $x > 1$ must be of the form:

$$\eta = \tilde{A} \sin\,(\ell y \mp \omega t)\,\ell^{-\sigma(x-1)} \; ; \quad x>1 \tag{3.4}$$

where $\sigma = \sqrt{\ell^2-\omega^2}$ and ℓ and ω must fit an dispersion relation obtained from (3.4) and the matching conditions (2.5):

$$\sqrt{\ell^2-\omega^2}\, Q\ (1;\omega^2,\ell^2) + \frac{dQ}{dx}(1;\omega^2,\ell^2) = 0 \tag{3.5}$$

For the limiting case $\sigma=0$ we have non decaying waves in the offshore region:

$$\eta = \tilde{A} \sin\ (\omega(y \pm t))\ , \quad x>1 \tag{3.6}$$

with ω fitting:

$$\frac{dQ}{dx}(1;\omega^2,\omega^2) = 0 \tag{3.7}$$

This equation is identical to (3.3) for $\phi = \Pi/2$, and its solutions are thus $\omega_n(\Pi/2)$. to explain the major importance of the resonance effect when ϕ is large we examine the limiting behaviours of (2.21) when $\phi \to \Pi/2$ for fixed ω. When $\omega \neq \omega_n(\Pi/2)$ Q_1' does not approach zero when $k = \omega \cos\theta \to 0$. Therefore $B,C,E \to 0$ and $A \to D$. In the near shore region all surface elevation vanish. Thus the run-up height tends to zero. On the other hand, when $\omega = \omega_n(\Pi/2)$ for some n, Q_1' is zero, in the limit $k = \omega_n \cos\theta \to 0$.

Equation (2.13) may be rewritten in the form:

$$\frac{d}{dx}(x\ \frac{dQ}{dx}) + [\omega_n^2(1-x)+k^2x]Q = 0 \tag{3.8}$$

Because k appears only in the term k^2, Q_1' must be expected to tend to zero as k^2. Therefore the ratio Q_1'/k approaches zero and $B \to 2A/Q_1$, $C \to 0$, $D \to -A$. In the offshore region we get a wave of type (3.1) and the run-up height becomes $2A/Q_1$. In table 1 we have listed some $\omega_n(\Pi/2)$ and the corresponding limits for F.

4. CONCLUDING REMARKS

Our results indicate that the run-up heights of periodic waves on a straight coastline do not vary considerably with the angle of incidence, ϕ, as long as this does exceed about 30°. If ϕ is close to 90° the picture is dominated by transition to trapped modes which occurs for discrete values of the frequency ω. When ϕ equals zero we have stationary inflection points for ω corresponding to eigenfrequencies for the near shore basin. For the intermediate values of ϕ these effects are less significant. These effects may have implication for sediment transport at beaches and may also be important for various techniques for utilizing wave energy. There are large amplifications near the shoreline for wavelengths much shorter than the length of the sloping part of the bottom. In these cases the amplitude of the incoming wave must be extremely small to avoid that nonlinear effects become important.

REFERENCES

Abramowitz, M. & Stegun, I.A., 1965. Handbook of mathematical functions, Dover Publications, New York.

Arntsen, Ø.A., 1978. Theoretical and experimental study of wave run-up on relatively steep slopes. Cand.real. thesis, University of Oslo. (Written in Norwegian).

Carrier, G.F., 1966. Gravity waves on water of variable depth. J.Fluid Mech. 24, 641-659.

Carrier, G.F. & Greenspan, H.P., 1958. Water waves of finite amplitude on a sloping beach.. J.Fluid Mech. 4, 97-109.

Gjevik, B. & Pedersen, G., 1981. Run-up of long waves on an inclined plane. Preprint Ser. no. 2, Dept. of Maths, University of Oslo.

Hall, J.V. & Watts, G.M., 1953. Laboratory investigation of the vertical rise of solitary waves on impermeable slopes. U.S. Army, Corps of Engrs, Beach Erosion Board, Tech.Memo no. 33.

Helal Badrawi, S., 1982. Etude théorique de la réflexion des ondes de gravité de grande longueur d'onde relative sur les plages. Thesis, L'Université scientifique et médicale, Grenoble.

Hibberd, S. & Pergrine, D.H., 1979. Surf and run-up on a beach: a uniform bore. J.Fluid Mech. 95, 323-345.

Kim, S.K., Liu, P.L-F & Ligget, J.A., 1983. Boundary integral equation solutions for solitary wave generation, propagation and run-up. Coastal Engineering 7, 299-317.

Kishi, T. & Saeki, H., 1967. The shoaling, breaking and run-up of the solitary wave on impermeable rough slopws. In Proc. 10th Conf.Coastal Engng. Tokyo, 1966. ASCE, New York.

Langsholt, M., 1981. Experimental study of wave run-up. Cand.real. thesis, University of Oslo. (Written in Norwegian).

Meyer, R.E. & Taylor, A.D., 1972. Run-up on beaches. in Waves on Beaches and Resulting Sediment Transport (ed. R.E. Meyer), pp.357-411. Academic.

Miles, J.W., 1971. Wave propagation across the continental shelf. J.Fluid Mech. 54, 63-80.

Pedersen, G. & Gjevik, B., 1983. Run-up of solitary waves. J.Fluid Mech. 135, 283-299.

Spielvogel, L.Q., 1976. Single wave run-up on sloping beaches. J.Fluid Mech. 74, 685-694.

Wiegel, R.L., 1964. Oceanographical Engineering. Prentice-Hall.

FIGURE CAPTIONS

Figure 1: Geometry of the model.

Figure 2:

Full drawn lines:

Relative run-up heights $F = R/A$ as functions of λ for $\phi = 0^{\circ}, 30^{\circ}, 40^{\circ}, 50^{\circ}, 60^{\circ}, 70^{\circ}, 80^{\circ}$. The curves are easily identified by observing that F is a strict decreasing function of ϕ.

Dotted lines:

Relative run-up heights for the two lowest eigenfrequencies plotted as functions of the corresponding wave lengths for ϕ ranging from 0° to 80°.

Table 1

n	1	2	3	4
ω_n	2.53	4.58	6.60	8.61
λ_n	2.48	1.37	0.95	0.73
F	4.06	5.06	5.78	6.37

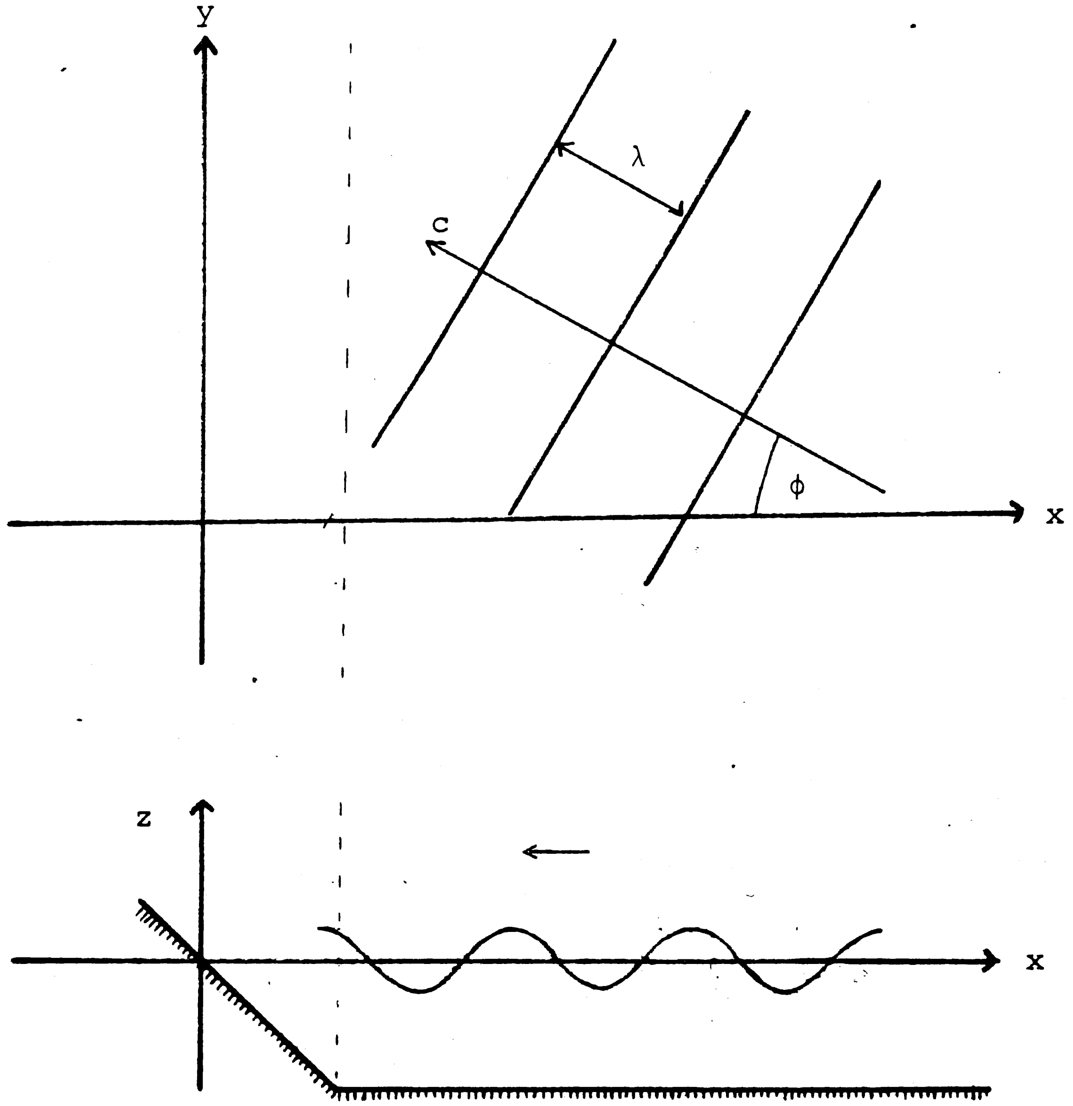

Figure 1.

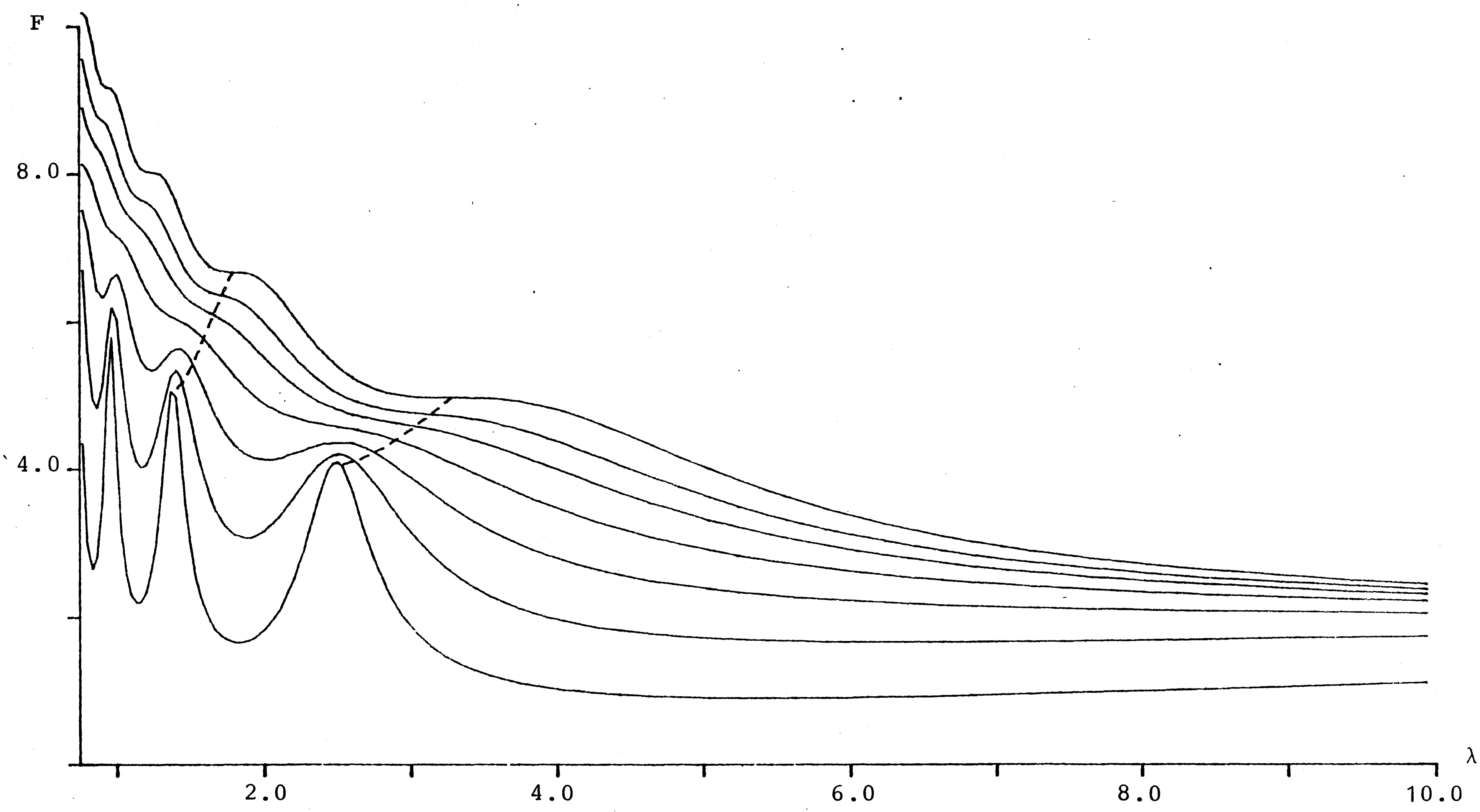

Figure 2.